MOTIVATING CATEGORIES: A SIDE EFFECT OF STRUCTURAL REALISM

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Abstract. In this paper I discuss Jonathan Bain's answer to the argument against radical ontic structural realism (OSR) based on the idea that a structure is an isomorphism class and thus cannot be the only thing that exists. I examine Bain's proposal of replacing the set-theoretic approach to OSR with a categorial approach and argue that several of his argumentative moves are deficient. First, Bain seems to define wrongly some of the mathematical concepts involved in category theory, for instance that of 'maximal ideal', and he also attempts to use these concepts in ways that would be detrimental to OSR itself. Both of these deficiencies undermine his claims. Second, the very form of Bain's argument is, to some point, self-defeating, since defining any category whatsoever presupposes some fixed set-theoretic framework.

Keywords: ontic structural realism, mathematical structuralism, category theory, set theory.

I. Introduction

The main thesis of ontic structural realism is that what exists in the world at the most basic level and is probed by the methods of science is structure – and the most radical version of that statement effectively says that this 'structure' can and must exist without any objects involved that would instantiate it.

A remark on semantics surely has its place near' the beginning of our discussion, as to not throw the potential reader into depths of confusion: the usual practice in mathematics and mathematical logic is to call 'structure' what is most commonly presented as a set endowed with additional gadgets like relations or operations or topologies. In philosophy that concept is more frequently denoted by 'structured set' or simply 'system', whereas the term 'structure' is reserved for the essence that is preserved under the appropriate notion of (iso)morphism. It is in the latter sense that the term shall be used here.

Even from this brief glimpse, we can see that a structure is something that arises from a richer construct, the system itself, and surely it would seem incoherent to bluntly state that it is the only thing that truly exists. For then, a structure would be an isomorphism class containing countless systems that satisfy the required axioms or properties, and those systems would also have to exist. This is the main counterargument brought up against the radical supporters of OSR, and this is what J. Bain seeks to overthrow in his 2013 article 'Category-theoretic structure and radical ontic structural realism'. The objection has its long history of circumvention attempts through logico-mathematical techniques like partially interpreted structures or Ramseification. However, Bain says that the counterargument is essentially correct, but it is based on a limited set-theoretic way of thinking, and by replacing it with one more 'categorial' or 'algebraic', one could actually see in a formal, mathematical way how structure can be thought of as primordial and dissolve the possible objections. My job will be to criticize the way he reaches that conclusion, but also to salvage what could be used for purposes that perhaps Bain did not have in mind.

II. Category theory

Category theory is a rather recent, but pretty controversial branch of mathematics (some would say that mathematics is a branch of category theory, a turn of phrase that is guaranteed to evoke its characteristic weirdness). It was created in 1945 by two mathematicians called Eilenberg and Mac Lane, in a paper of which they were sure it would be the last ever written on the subject, to give meaning to some terms like 'natural transformation', which were used only in an informal way when stating theorems in algebraic topology.

After several decades in which new developments were brought both into the subject, like Kan's adjoint functors or Lawvere's elementary toposes, or out of it, like Cartan and Eilenberg's recasting of homological algebra or Grothendieck's of algebraic geometry in category-theoretic terms, the subject proved that it was here to stay. Nevertheless, its highly abstract character and unintuitiveness doomed it to become only a niche interest to the majority of mathematicians.

In a way, the idea to replace sets by categories is not new. The first serious attempt was the 1965 one by Lawvere, called the 'elementary theory of the category of sets', in which he tried to provide an axiomatic set theory expressed in the language of categories. The topic has gone in and out of fashion since that time, culminating with the grand ambitious project du jour that is homotopy type theory. What is interesting about Bain's proposal is the solid philosophical foundation of structural realism that lies behind it, though, as we shall see, it is not so clear whether the project of bringing mathematical meaning to it can stay faithful to that goal.

III. Bain's arguments

Bain's actual arguments exploit the notion of 'universal property' that is familiar to category theorists. It is obvious that we can define only one function to a set with one element, and as many functions from a set with one element as there are elements in the projected co-domain of our functions. This gives a characterization of the elements of a set A as the functions that have as the co-domain the set A and as the domain the set to which only one function can be defined from any other set (called a terminal object), i.e. a definition in terms only of sets and functions, withholding any assumptions on an 'internal structure' that the sets may have a priori.

Even though I am a strong supporter of using category-theoretic concepts when doing mathematics, I cannot but point out the fact that this sort of argumentation misses the spirit of structural realism. Of course, the objects get thrown away in the process, but why can we say that the 'relations' or 'structure' gets preserved? The whole edifice of a structured set is overthrown, and the only entities that may count as relations in a category are the morphisms, who are of a relational nature, but not between the original objects, but between the sets or structured sets or more generally containers of objects, and who take the name of categorial 'objects' that can only be thought of as a cruel coincidence.

Or perhaps not so: a category defined as a class of objects together with a class of morphisms such that so and so axioms hold is not particularly revolutionary from the way set-based systems like groups or rings are defined. The objects are rightly named, the only victory we have achieved is that we have moved up a higher level of abstraction. And it is not a hollow victory, for the crystallization of concepts like terminal objects and elements-as-morphisms can surely give us precious insights when we move to a category that does not resemble so much the one of sets.

And this movement is what Bain does in the next section of his article – he highlights the equivalent reformulation of general relativity through 'Einstein algebras' instead of Lorentzian manifolds and tensors. This is part of a grander mathematical phenomenon called 'algebra-geometry duality' in which geometrical spaces may be studied through algebras of functions of them, or, more sophisticatedly, through sheaves of functions, as he proceeds to show us. The example, however, breaks down in more than one point.

Firstly, he supposes that the points of a space are the objects that we have to dispense with, and all other notions are fair game as long as they are reformulated as corresponding enhancements on the algebra. This is highly contrary to what structural realism deals with, which is unobservable and controversial 'objects' like forces or Lagrangians, which escape largely unscathed from the tumultuous algebraic transfiguration.

Let us suppose, however, that this is simply an analogy, that we do not have to take it literally, and that points are what we are after in the (toy?) example. Bain supposes that we might recover them from the maximal ideals, for which he gives two wrong definitions, one which is actually that of a maximal subring, and another which simply states that they are elements of an algebra. Still, it is known that maximal ideals (properly defined!) contain the information necessary to recover the points from an algebra and hence reconstruct the dually equivalent geometrical category. However, this only means that the whole theory of general relativity has two interesting categorial models, none more valid than the other. As the view of Hilbert (the one which Landry establishes in her 2012 article 'Methodological Structural Realism') shows us, 'it is not that theories come without interpretations, it is that they come without fixed interpretations' (Landry 2012, p. 38).

Finally, he brings sheaves into the picture. Now, sheaves have indeed played a large role in the rigorous development of the duality considered above. But Bain only talks about sheaves defined on a topological space, which even as they contain as many algebras as open sets are in the space, they are founded on a geometric nature and so their whole existence goes against his point. A more refined, steelmanned argument could exploit the idea of sheaves defined on a point-less 'site', a notion introduced by Grothendieck to help in proving the Weil conjectures. Or, even better, toposes of a more geometrical character. But that is a story for another time.

The elephant in the room is, however, that any category that we may define presupposes some set-theoretic framework. We cannot talk about the category of groups until we know what a group is and what the building blocks from which we forge it are. The answer, as foreshadowed above, is to axiomatize the idea of a category of sets or of another type of structure and work only with the axioms. This is what Lawvere and others did successfully. However, this raises the question: what can then we declare that 'exists', given that the only thing of which we can be sure of is finitist syntax and in a sufficiently expressive syntax any conceivable structure may be bootstrapped from scratch?

IV. Conclusion

The answer is that we have made a confusion along the way between structural scientific realism and mathematical structuralism. We are not here to answer which mathematical foundation is more 'fundamental' – only which of them can accurately represent physics and structuralist ontology at the same time. And although Bain's arguments fail to be completely accurate, it seems likely that a physics based on category theory could implement some day the philosophical ideas of structural realism – he gives, for example, small steps taken by Baez et al. And even if not, structural realism is a worthy motivation that may provide the required momentum for a definitive re-grounding of mathematics through category theory. It would be like the case of Frege: his logicist program ultimately failed, but he gave us first-order logic and new insights that could be obtained through that instrument. This may be a view motivated by pragmatism – nonetheless I think it is virtuous to seek to salvage all the good that may show up from such an endeavour.

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