THE NATURE OF THE STRUCTURES OF APPLIED MATHEMATICS

CĂTĂLIN BĂRBOIANU

Introduction

The current orientation toward the structural model of science is still in full impetus, not only in metatheoretical analyses, but also in the constitution of science itself as the object of these analyses. Such an orientation has had as a motivation neither an alternative to the previous logical-empiricist and sentential models nor the domination of structuralism in the analytical philosophy of the last century. Rather, it is motivated by a set of epistemic criteria that became normative in the philosophy and evaluation of science in the last 40-50 years: a) Explanation (along with prediction) became the main function of science, and the epistemology and philosophy of the science of the last decades had as their central theme explanation and a theory of explanation. In the context of this focus, the explanatory contribution of the classical (set-theoretic) structures in science could not be neglected, even though it is the subject of intense ongoing debate. b) The constitutive use of classical and mathematical structures in sciences exhibits the potential of categorial unification, intertheoretic and interdisciplinary connectivity, and applicability. c) The structural fundament generates a predisposition not only to the advanced mathematization of sciences, but also to a metatheoretical justification for the use of mathematics. Mathematics not only stands as a method, but is also a constitutive part of science, due to the mathematical structures which can be connected with the investigated structures through various methods, like direct relationing, correspondence, or interpretation.

A structural theory is one within which the relations between parts or objects prevails epistemically over the nature, individuality, and specificity of the latter. These relations are described and defined in a language compatible with formal logic, thereby allowing the classical logical operations between statements and the formation of classes of models of a theory. Such compatibility generates a structural extension of a logical type and a sort of continuity with the sentential model of theories. Adopting non-formal set theory as a universal language for the empirical sciences yields a continuity of language and of conceptual analysis used in the metatheoretical models and also provides the structural theories with a uniform sense – that is, a universality in which the notion of structure becomes essential, distinguishable at several levels, and having different natures. revealing four types of structures: 1) the logical structures of the scientific methods and 2) the set-theoretic relational structures (either conceptual or physical-empirical) that are created and extended through procedures specific to set theory and universal algebras, as well as 3) the mathematical entities/structures participating in scientific theories still in the form of structures (this time mathematical structures in Bourbaki sense), and even 4) the internal structure of a scientific theory itself[1]. The continuity and universality specific to the structural approach has the potential of apparently unproblematic moving to the metatheoretic investigation by linking the first-order concepts with those of a superior order through the same methods, and also the object of the theoretical investigation with the investigation itself, thus integrating them all into one structure.

A particular case of structural metatheoretical investigation, but central in the contemporary philosophy of science, is the philosophical problem of the applicability of mathematics in natural sciences. In this paper, I argue that in the structural

---

[1] In the sense of constructive, not of theoretical content.

metamodels of application and applicability of mathematics, the reductionist structural approach based on a set-theoretic concept of *relation* of a primary mathematical "type," cannot prevent putting in evidence structures of different natures and epistemologies. Moreover, some structures are incompatible with the classical ones, with respect to certain epistemic principles specific to the applicability of mathematics. This diversity of natures and this incompatibility render problematic the *justification* (at the metatheoretical level) of the use of mathematical modeling as a main method of scientific investigation. Such justification should be, along with representation, an essential function of any theoretical model of the applicability of mathematics.

In the first section, I present a conceptual analysis of Bourbaki mathematical structure with an eye toward the classical set-theoretic concept of structure. Such an analysis brings back into discussion the problem of the empirical or non-empirical, formal or non-formal nature of the concept and continues with a brief presentation of the contemporary structural models of application of mathematics, for which I identify a double nature (theoretical-applicative and metatheoretical). In the second section, I argue for the epistemic character of the suprastructure created through the structural metamodel, which integrates the source and target structures as well as their external relation. I then show that the epistemic nature of the suprastructure is incompatible with its set-theoretic homogeneity with respect to both nodes and relations, and this incompatibility does affect the justification function. Finally, I argue that if we give up the idea of suprastructure and keep the homogeneity of each of the two corresponding structures, the difference between their natures leads to a problem of truth, which in turn does affect the justification function of the metamodel.

I. Structural application and applicability of mathematics

A first semantic aspect of the common concept of application of mathematics in sciences is *import*. Application of mathematics assumes a multifaceted import consisting of the methodologic import, the conceptual one, and also that of the necessary mathematical truths. During the history of science, such a general application has shaped three classical roles of mathematics, namely the constitutive role (for scientific theories), the descriptive role, and the inferential role. In this rough conceptual framework, the applicability of mathematics (as a property) would mean the possibility of performing these roles, but also the success of this practice. A refined definition of applicability of mathematics, subsumed to an adequate conceptual and theoretical framework, is currently a target of the contemporary philosophical accounts of applicability of mathematics.

I.1. The structures of pure mathematics, unity, and applicability of mathematics

In *The Architecture of Mathematics* [1950], N. Bourbaki does not aim at defining the concept of mathematical structure within a pre-established, perhaps formal, theoretical framework, nor at developing a theory of structures[2]. Rather, the description starts from an overview on mathematics as a discipline by asking whether the autonomous theories emerging and developing within mathematics – thus becoming separate (through their goals, methods, and even language) – do affect the unity of mathematics. Bourbaki's answer to this question (which was not new at all) is negative, and the argumentation is made around the concept of mathematical structure.

For Bourbaki, it is obvious that logical formalism is a system of logical rules adapted to the mathematician's need to build a theory as a concatenation of propositions derived from each other. This convenience cannot ground a unifying principle for

---

[2] Bourbaki describes the concept of mathematical structure of a certain type even before the cited work that is in Volume 1 of the *Elements*, in terms of set theory.

mathematics. In addition, the deductive reasoning through syllogistic enchainment, also specific to any non-mathematical discipline, is no more than a transforming mechanism applied to a set of premises, and as such it cannot serve for the characterization of those premises nor of the complexity of the various mathematical theories.

Even though logical formalism and axiomatic method seem to provide the linking element for the unity of mathematics, Bourbaki argues, these two cannot establish this unity, just as physics and biology, for instance, cannot be unified just on the basis of the experimental method or hypothetic-deductive method that they both use. What these methods cannot provide is an "intelligibility" of mathematics, which will be responsible also for the unity of this discipline; instead, the concept of mathematical structure would ensure, in Bourbaki's view, this unifying intelligibility.

That said, the concept of Bourbaki structure does not belong to a formal system and is not metamathematical but is generated from a complete perspective on the content of mathematics. The basic idea is to extract the smallest number of independent properties (of the *relations* between the elements of a set from a mathematical theory or domain) from which any other property can be derived, and to ascertain the applicability of these properties to the relations between the elements of other sets (including sets from other theories), provided that the nature of these elements does not in any way influence the *derivation* of that property. Ignoring the nature of the elements is crucial in defining the concept of mathematical structure, which thereby appears as a concept applied to the sets of elements of an unspecified nature. Mathematical structure is not given *a priori* but is defined on the basis of certain pre-established relations, for which specific conditions or properties are formulated explicitly as the axioms of that structure. Developing the axiomatic theory of a structure means deducing the logical consequences of that structure's axioms by excluding any hypothesis regarding the elements that stand in a relation (including their nature).

Next, Bourbaki identifies three primary types of mathematical structures (called 'mother-structures') by the nature of the relations established in the base set, namely, algebraic structures (corresponding to the relations in the form of laws of composition), order structures (corresponding to order relations), and topological structures (corresponding to the set-theoretic relations of topological type). These three types are recognizable in existent mathematical theories, in singular or multiple form. (There are structures whose sets of axioms are specific to more than one of the basic types; for example, the structures of algebraic topology).

In Bourbaki's view, this concept of mathematical structure and this typology (admittedly exhaustive) are the premises that enable the axiomatic method to validate the unity of mathematics, which can justifiably be called a *structural* unity, given its set-theoretic fundament with interconnectivity potential. Evidentiating a concept common to several mathematical theories (at the level of the relations defined within these theories and independent of the language and methodology of those theories) grants them – according to Bourbaki and his followers – an intelligibility that the axiomatic method and necessary truths cannot grant by themselves.

Written in a period when mathematized physics was at its peak and particle physics was in full advent of its discoveries "driven" by mathematical formalism, Bourbaki's work underscores the unexpected capacity of mathematical structures to adapt to the problems of physics and to contribute to its theoretical content. This contribution or constitutive applicability remains fruitful with the increasing complexity of the mathematical structures through formal procedures that assume emptying the empirical-intuitive content of the axioms of the mother-structures. This applicative nature of the structures appears as remarkable and unexpected. From a philosophical perspective, it seems that there is a preadaptation of mathematical structures to physical reality, which assumes an intimate relation of mathematics with empirical sciences, a relation that seems to be more "hidden" than accepted *a priori*. By anticipating the challenge, ten years hence, of Eugene Wigner [1960], known as the syntagma "unreasonable effectiveness of mathematics," Bourbaki launched an indirect and unformulated challenge to philosophers of mathematics and science, limited to his concept of mathematical structure: The adaptability of mathematical structures to the theoretical content and problems of empirical sciences, admittedly unexplained, seems to contradict the *formal* procedure generating these structures, since this procedure relies on emptying notions and axioms of any empirical content or influence. On the other hand, mathematical structures became not only applicable, but constitutive to natural sciences, especially to physics, where a mathematized branch assumes not only a mathematical methodology, but also a theoretical framework founded on mathematical structures. Under these circumstances, Bourbaki asks whether the unity of mathematics is the outcome of formal logic or simply this scientific fertility. In other words, the unity of mathematics is not one of an inert structural skeleton but one of a more complex organism in evolution with the scientific environment that influences the organism through mutual exchanges. Are mathematical structures abstract, inert forms or do they have a certain "life" consigned by their applicability? Can we somehow have the certitude of this applicability in the future course of the evolution of science? Nowadays, these questions have been reformulated, decomposed, and refined within the problems of philosophy of applicability of mathematics, which has developed as a delimited field of philosophy of science beginning in the 1990s around the so-called 'miracle' of applied mathematics. The nature of the mathematical structure in relation with its participation in constitution and problems of empirical sciences has remained a subject of debate as enduring as ever, and this debate may shed some light on several issues related to scientific 'miracles' and the success of structural science.

I.1.1. The empirical component of mathematical structures

Intelligibility of structural mathematics to which Bourbaki refers increases when we assign an empirical influence or interpretation to the axioms of the structures.

This empirical component of the axioms, which is transmitted to the structure and then to the mathematical theory that employs that structure, is independent of any mathematical ontology and in no way is it subsumed into a Millian-type mathematical empiricism or an even weaker one. The same independence can be noted when we place the empirical component within an interdisciplinary framework, on the basis of the idea that human action of mathematical creation cannot be analyzed exclusively philosophically. It is worth mentioning the advances that have been made in the new field of perceptual mathematics (see especially [Lomas, 2002], [Teissier, 2005], [Ye, 2009] and [Mujumdar & Singh, 2016]), which, once articulated in a clearly crystallized theory, will pose serious problems to any debate either supporting or opposing mathematical empiricism, as long as the tools of that predicted theory are multi- or inter-disciplinary.

Of course, there are mathematical structures elaborated with no empirical influence, some of them even resulting from an axiomatic intellectual game. Even though they do not have an empirical origin, such structures could eventually find their applicability in a future mathematical theory based on originally empirical structures, or to an empirical science, thus acquiring indirectly an empirical component[3]. Moreover, if the idea of acquisition of the empirical component through internal or external[4] applicability is rejected, Bourbaki's canonical typology of the structures still ensures the empirical component, through the fact that those basic structures are present within the complex

[3] So many times during the history of science, a structure or mathematical theory created independently of any empirical-scientific problem finally found its successful application in science, and this fact is also one of the premises of E. Wigner [1960] in qualifying the successful general application of mathematics as "unreasonable." On the other hand, even one of the pretended solutions to Wigner's problem (called in the literature "empirical-origin solution") invokes interconnectivity of the mathematical concepts, among which those having an empirical origin create the link between the two universes of different natures – mathematical and empirical. For a succinct description of this solution and a well-organized presentation of the classical solutions proposed for solving Wigner's problem, see [Bangu, 2012, p. 135-143].

[4] With respect to mathematics.

structures, and each type of mother-structure reflects an obvious empirical influence at the level of representations, actions, phenomena, or objects as follows: the laws of composition reflect counting, collecting, adding, multiplying, composing, etc.; order axioms reflect quantitative and magnitude order, comparison, hierarchy, etc.; topological axioms reflect spatial surrounding and vicinity, isolation, proximity, form, limit, and continuity.

Generally, the structures of elementary mathematics reflect human experience, and this fact became paradigmatic. Mathematics starts from the experience of human activities[5] by creating its primary structures through the abstractization of this experience and extending them through logical formalism and the axiomatic method.

I.1.2. Mathematical structure: formal or non-formal?

The empirical component of the typological concept of mathematical structure (Bourbaki) seems to "alter" the formal nature of the concept described through the necessity of the logical-formal treatment. Even though the extension of the concept as an equivalence class or category seems to free it from this empirical component, question remains open as to whether its nature is formal, non-formal, or somehow mixed. On this theme, we could draw a parallel with the Fregean view on the semantic applicability of mathematics, which is based on second-order predicative logic: For Frege, mathematical statements are not statements about physical objects, but about conceptual extensions of concepts (such as classes, properties, etc.) regarding those objects. Mathematics does not state laws of nature, but rather, laws of the laws of nature; mathematics cannot be applied to an empirical context, but to a thought about an empirical context. Meanwhile, the logic of mathematical propositions remains formal, even if we can assign to them indirect references[6]. By ignoring the

[5] Refer also to the empirical Babylonian mathematics, in studies of history of mathematics.

[6] For an overview of Frege's work on the nature of mathematics and its entities, focused less on the aspects of his logicist construction and more

logicist specificities of the Fregean model and its platonist orientation but keeping the analogy, we can say that mathematical structures such as classes and extensions have a mere formal nature; however, this characterization can be easily rejected if considering Bourbaki's terminology, containing both formal and non-formal terms such as 'axiomatic *method*' as a structure-generating tool.

In a critical interpretation elaborated around the concept of the Bourbaki structure, L. Corry [1992] advocates for the dual formal and non-formal character of the concept. Corry claims that the so-called structural character of contemporary mathematics reflects clearly a way of *doing* mathematics, which can be described in non-formal terms despite attempts at creating a formal theory within which the non-formal idea of a mathematical structure is elucidated in a mathematical way.

Even though, as Corry argues, the dual formal/non-formal nature of the mathematical structure does not pose methodological and epistemic problems to the practice of pure mathematics, once we pass the border into the domain of applied mathematics, the unresolved issue of the exact nature of a mathematical structure raises fundamental problems to the philosophical accounts dealing with the applicability of mathematics.

I.2. The structures of applied mathematics: mathematical, classical, epistemic, pure, or mixed

In structural sciences and standard applications of mathematics[7] in sciences and everyday life, the rationale of description, representation, interpretation, and inference lies in the notion of structure. This primary concept of structure does not belong to any complex

on the problems of ontology, philosophy of applicability, and language of mathematics, see [Dummett, 1991].

[7] I shall call *standard* application of mathematics an application submitting to the scheme of the structural model of application of mathematics, based on the structural analogy via morphisms of structures, in the sense of Bueno and Colyvan's [2011] inferential conception of applied mathematics.

formal system but is described as a network of objects/nodes/ positions connected through relations. These objects have an atomic status, and the relations between them are conventional or criterial associations. The epistemic prevalence of one set of relations or set of objects over the other within the structure defines the type of structuralist or respectively non-structuralist perspective from which the structure is considered, but this aspect counts only at the level of philosophy of science and not as concerns science itself and its outcomes, which have been confirmed regardless of the theoretical nature or ontology of science's objects or entities.

Thus, the primary concept of structure reverts to the primary concept of *relation* as an ordered association of some given objects, in a trivial set-theoretic sense – namely an ensemble $S = \langle D, R \rangle$, with D non-empty set and $R = (R_n)_n$ a family of sets of ordered n-tuples of elements from D ($R_n \subset D^n$). This primary concept – I shall hereinafter call it *classical structure*, corresponding to the sense of static structure of Resnik [1997, p. 202-209] – grounds science and structural knowledge, and it is difficult to say whether its set-theoretic nature (under non-axiomatized set theory) is sufficient to justify the attribute of 'mathematical', as long as it seems to be more of a mental-psychological concept. The brain neurophysiology of humans as well as other species includes *association* as a basic process, either as random association (imagination and convention) or on the basis of criteria established through observation and perception (pattern recognition, using previous knowledge, etc.). This fact raises the question of whether the nature of the relations within a classical structure (and implicitly the nature of that structure) is mathematical (set-theoretical) or mental; in the latter case, the association through the symbolism of the parentheses would have only the role of a merely conventional "transcription." On one hand, the question is important when we consider the problem of applicability (including the constitutive one) of mathematics in structural science, for this general process means connecting and mutually integrating structures from domains of different natures, which apparently implies the necessity of a common nature of those structures. On the other hand, the question loses its import when acknowledging that we haven't at our disposal a defined concept of "mathematical relation" – we know only that is constitutive to the notion of mathematical structure. Therefore, it is legitimate to ask whether the classical structures can be identified with the mathematical ones, with the reservation of a problematic acceptance of the same set-theoretic nature for the primary concept of relation in both concepts.

The literature on classical structures is developed more regarding structuralism, structural realism, and ontology of structures[8] and less toward the epistemology of structure with respect to structural mathematized science. As concerns the roles of mathematics in structural science and the way mathematics plays these roles successfully, the epistemological aspect prevails over the rest, because the structures are used in a functional mode, becoming tools or methods of acquiring knowledge. In what follows, I argue that the two distinguished types of structures – mathematical and classical (set-theoretic) – have different epistemologies when participating in mathematical modeling within empirical sciences, even under the hypothesis of a shared set-theoretic nature. As a preamble, I shall present in brief the structural model of the standard application of mathematics, focusing on the nature of the structures involved.

I.2.1. The primary structural model of application of mathematics

Pincock [2004] renders central the idea that application of mathematics assumes an *analogy* between mathematical structures and certain structures of the physical universe obtained through idealization; the concept of structural analogy is represented by

[8] See, for instance, influential works such as [Piaget, 1968], [Shapiro, 1997, p. 71-108], [Ladyman, 2007], [French, 2014], [Arenhart & Bueno, 2015].

the set-theoretic notion of homomorphism or isomorphism[9], as a structure-preserving application between two different domains[10].

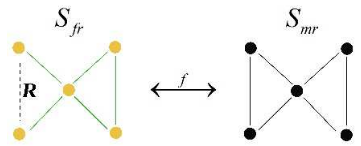

Fig. 1. The primary structural model of application of mathematics

The motivation for such representation consists, on one hand of strengthening the uniform semantics of the mixed[11] statements of applied mathematics, and on the other hand, of inducing a minimal inferential character to the mathematical modeling, which is ultimately a means of acquiring new knowledge in the empirical domain. This inferential character was missing in the 'internal relation' models'[12], among which Frege's semantic applicability had solved the problem of the uniform semantics in a simple and unobjectionable way.

The homo/isomorphic function f is an external relation between the two domains (assumed to exist *a priori*) which forms the correspondence between the nodes of the structures. The two structures – the physical (S_{fr}) and the mathematical one (S_{mr}) – are constituted through a process of extraction from a larger

[9] Depending on each particular application.

[10] Although Pincock is granted as the author of the theoretic model based on structural morphism, there are references to this model in previous works, such as Baker [2003], Balaguer [1998, p. 109-112] or Leng [2002].

[11] Containing both physical and mathematical terms.

[12] The typology of theoretical models of the application of mathematics by the nature of the relation established between the physical and mathematical domains also belongs to Pincock [2004]. An 'internal relation' is actually an identity criterion: An internal relation is a relation in which an object must stand in order to be that object. The immediate example is set membership, which stands as a relation between a set and any of its elements. A relation that is not internal is external. Frege's model of semantic applicability, as well as its set-theoretic analog developed by Pincock [2004], is an 'internal relation' model.

structure: the physical structure puts in evidence only the physical objects and relations (connections) between them that are *relevant* for the application (descriptively, but also making possible an analogy with a mathematical structure *conveniently* chosen); the mathematical structure consists of those theoretical parts from within pure mathematics whose results (derivations) will be effectively engaged in the application. The first extraction is what we call the *idealization* of the empirical context, which in fact is a double idealization – that of the extraction/isolation from a larger system and structure, then that through which physical objects are granted the status of nodes of the set-theoretic structure by canceling their physical complexity.

A set-theoretic reduction is also operated in the mathematical domain because initially the structures being prepared for engagement in modeling are those in the classical sense of a Bourbaki structure. The three types of mother-structures, described through sets of axioms free of empirical content and linked through the axiomatic method, can be described set-theoretically as relational structures[13] in an unproblematic way. Such a reduction is not an idealization (of the type of the physical one), but rather an equalizing, because the lack of empirical content of the structure's nodes ensures their atomic status, and the relations of the structure are *defined* or *derived* mathematically.

In such a theoretical model, application of mathematics works by inferring an unknown connection (relation) R in the physical structure on the basis of the homo/isomorphic character of function f. In the mathematical structure S_{mr}, all relations are known, either as definitions or logical derivations. If nodes $f(x)$ and $f(y)$ stand in a relation in S_{mr}, then x and y stand in a relation R in S_{fr}, which was not known before the application. The inferred relation R is then interpreted in the mixed language of the empirical context, thus contributing to its description.

[13] Even if algebraic structures involve operative relations.

Pincock does not develop further a theory of an *external* relation between the mathematical and empirical domains and does not advance a formalism of the corresponding structures or of the structural extraction. The structural formalism is approached by Bueno and Colyvan [2011] in their theoretical model called 'inferential conception of applied mathematics' (ICAM).

I.2.2. The inferential conception of applied mathematics

Even though it is an extension of the Pincock's primary 'external relation' model, ICAM is not merely structural exhibiting some pragmatic context-dependent characteristics of the process of applying mathematics. The core principle of ICAM is that the fundamental role of applied mathematics is inferential (even though the functions of a mathematical model may be multiple), and this role ultimately depends on the ability of the model to establish *inferential* relations between the empirical phenomena and mathematical structures. In terms of Bueno and Colyvan, ICAM consists of a three-step scheme:

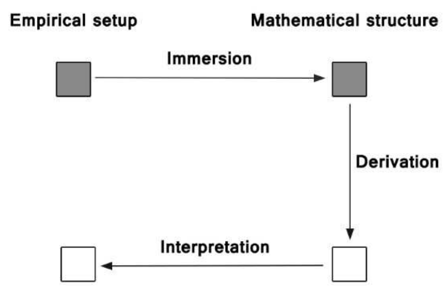

Fig. 2. ICAM scheme

1. (Immersion) establishing a homo/isomorphic function from the empirical context to a convenient mathematical structure through which to link the relevant aspects of the empirical situation to the appropriate mathematical context[14].

[14] This function is not unique, and choosing the right one is a contextual problem in the charge of the mathematician, depending on the particularities of the application.

2. (Derivation) Deriving the consequences through mathematical formalism within a specific mathematical theory, by using the mathematical structures chosen at the immersion step.
3. (Interpretation) Interpreting the consequences obtained at the derivation step in terms of the empirical context by establishing a homo/isomorphic function from the mathematical structure to the initial empirical context[15].

In this theoretical framework, the primary structural model is present at both the immersion and interpretation steps. The aspects of surplus of structure – both in physical and mathematical domains – remaining outside the process of mathematical modeling[16] are assimilated through the introduction of the notions of partial structures, then through partial homomorphism/ isomorphism[17]. The partial nature of relations and structures reflects formally the incompleteness of our knowledge about the investigated physical domain and has an epistemic character rather than an ontological one. The distinction between the immersion and interpretation steps allows different external functions to operate independently within the model; these functions represent the possibilities of revising the structural arrangement, of refining the idealizations, and even of formulating new problems related to the original problem, including new empirical discoveries. These processes actually take place in the applied-mathematics practice.

The entire structural approach of the application of mathematics to an empirical context, from the primary model to ICAM (also known as 'mapping accounts'), is based on the idea of

---

[15] This function is not necessarily the inverse of the immersion function, although in many concrete situations it can be.

[16] The former, through the idealization of the physical system, and the latter within the derivation step, which assumes the selection of the convenient mathematical structure from the larger mathematical context.

[17] This formalism of partial structures and morphisms was developed before the work of Bueno and Colyvan [2011], in [French & Ladyman, 1998, p. 51-73], [Bueno, French, & Ladyman, 2002] and [da Costa & French, 2003].

a formalized analogy between a domain of mathematical structures and a domain of conventional-relational, so-called physical, structures This analogy is identified through the particularities of the application, but also through the established analogy itself, as part of the actual application process. Such collaboration and interdependence between the source and target domains and their external relation (the morphism function) allow the structural model to represent theoretically not only the application of mathematics, but apparently also (to a certain degree) its applicability.

I.2.3. Functions and nature of the structural metamodel of application and applicability of mathematics

There are at least two general targets of investigation of the problems of applicability of mathematics, for it is not only the success of applied mathematics that stands as an object of the research, but also the general use itself of mathematics as a method of scientific investigation. As an epistemic-intellectual responsibility, this use needs to be justified in the given conditions, *i.e.*, differences of ontological, epistemological, and logical natures between the source and target domains, as well as the existence of the unsuccessful applications. Thus, any theoretical model developed for solving the problem of applicability (including the structural ones) should have two essential functions: representation (of the processes of application of mathematics) and justification (of the application), – in other words, establishing a general applicability of mathematics independent of any circumstantial factors of a practical-theoretical or methodological nature.

At the beginning of the last decade, philosophy of mathematics was still dominated by debates regarding the explanatory role of mathematics versus its representational role, explanation being the central theme of philosophy of science and epistemology. This "explicative" trend has also influenced to some degree the motivation for the creation of the structural models of application and applicability of mathematics (especially ICAM), whose authors adapted to this trend with an eye on Wigner's problem. Thus, we can view a structural model of application and applicability of mathematics from two perspectives, revealing its two different natures: on one hand, a general model of reasoning based on pure mathematics and applicable to any applied-mathematics problem, reverting through instantiation to the classical concept of mathematical model dependent upon the particularity of the application (the theoretic-applicative nature); on the other hand, a universal model representing every possible application, the general process of applying mathematics by using its structures, but also the structural correspondence between the two domains, that are established through the application as well as that assumed *a priori* (the metatheoretic nature). In the theoretic-applicative model, explanation (as an important function specific to a dominant category of applications) remains at the first level, that of the instance of application, In the metatheoretical model, explanation also appears at the second level in the form of (metatheoretical) justification of the general use of the mathematical method. By specifying the two natures of the structural model, we can see that the theoretic-applicative model represents the application of mathematics while the metatheoretical model represents application *and* applicability of mathematics.

The justification for applying the mathematical method as a function necessary to an adequate theoretical model of applicability of mathematics manifests itself through the metatheoretical nature of the structural model.

II. Different natures of structures in the structural models

In section I.1.2, we revisited the debate on the nature of the mathematical structures themselves as being formal or non-formal with respect to the empirical content incorporated through the constitution of the primary types of Bourbaki structures. In the current section, I shall extend this discussion to the *suprastructure* that is created with the finalization of a mathematical application representable through the structural model.

In the primary model, as well as in the ICAM scheme, we deal with processes of *constitution* of the structures: on one hand, the constitution of the structure from the empirical context through the associations as relations of the objects conveniently establishes a further analogy with the mathematical structure or structures participating in the inference; on the other hand, the correspondence established between the source and target structures creates in turn a structure completing the two existing structures and thus creates a suprastructure. Since functions and their compositions are representable set-theoretically, the suprastructure thus created is also of a classical set-theoretic type and includes the participating mathematical structures.

The question arising is whether, when qualifying the nature of these structures, we should consider only the set-theoretical aspect, or to consider also the intentional aspect, which is justified by the goal- and user-dependence of the mathematical model. Regarding intentionality, there is a conventional character of the structural arrangement in the target (empirical) domain – the physical objects do not stand in defined relations, as is the case with mathematical structures, but they are put in relations through a mere conventional association (as *n*-tuples), motivated through criteria of convenience[18]. A conventionalism can be also assigned to the mathematical structure at the level of creation of its axioms, but this one is of a very different nature – by having a justification based more on relevance than convenience. This latter conventionalism could be also dissolved if we consider it against the necessity of the integration of the mathematical structure into an existing logical system (the mathematical theory within which the created structure will function). In conclusion, the only difference in nature of the mathematical structure and the classical

[18] Moreover, the structural arrangement of the target domain may be modified as the result of testing the mathematical model (even the ICAM scheme allows this change), which strengthens the conventional character of the arrangement.

structures participating at the structural mathematical modeling could result from the different conventionalism of the two types of structures at the intentional level. If this distinction is accepted, we call the classical structures of the empirical domain and the suprastructure created through the theoretical model of application *epistemic* structures since they are constituted with the goal of acquiring knowledge in the form of partial relations inferred through means of the mathematical model. In this sense, mathematical structures will have an epistemology different from that of the classical ones, and this epistemology is "located" at the level of the *relations* of the structure.

We have a difference in nature of the nodes of the structures also. The structures created in the empirical context assume sets of *physical* objects, so that we are forced to accept the concept of a set (as mathematical or set-theoretic notion) having physical objects as elements. The entire structural scheme of application of mathematics relies tacitly on this principle, which for M. Steiner [1998, p. 22] and Pincock [2004] is unproblematic[19]. The necessity of keeping the set-theoretic nature of a set with physical objects despite its physical content is created by the presence and functionality of the structural morphism, which is the core concept on the basis of which the model works and is defined in set-theoretic terms.

In another paper [[name deleted to maintain the integrity of the reviewing process], 2017, p. 80-87], I argued that such a concept does not meet the two criteria widely acknowledged as adequate for a concept of set – namely, individuality (a set must be a set of terms, objects, elements, as individuals) and plurality (a set must be characterized by a diversity of individuals, not only in a logical-numerical sense, but also as an identity criterion). As concerns individuality, I have shown that the unavoidable systemic interaction of the physical objects in sufficiently large

[19] The principle of a set of physical objects is also essential for the 'internal relation' account sketched also by Pincock [2004], in which applicability is reduced to the set membership relation.

systems poses problems to the constitutive differentiation of the objects. In other words, delimiting the physical object for individualization can be done only spatiotemporally, while the set, as a mathematical concept engaged in the practice of pure mathematics, is non-temporal. Temporality can affect the plurality of a set as well as the systemic relationing/interaction. Thus, a set can be only a set of atomic references of those physical objects, and individualization through reference – an intentional and conventional process – can be done only in case of the elements of "pure" sets, through the mathematical definition. In addition, for sets containing both physical and mathematical objects, I have argued [name deleted to maintain the integrity of the reviewing process, p. 87-90] that the diversity (as non-identity, not non-individuality) of their elements cannot be tested for the sets that are described predicatively.

Despite the above arguments, accepting the principle of a set of physical objects at the conceptual level cannot prevent an immediate differentiation of the epistemological nature of the types of structures participating in the structural model: The structures of the empirical domain will have objects with empirical content as their nodes, which eventually interact with each other in systems outside the structure, while the structures from the mathematical domain will have nodes with an atomic status, but with no empirical content. The fact that elements of "purely" mathematical sets may participate in mathematical theories outside the structure is not an analogue of the systemic interaction from the physical domain because in the mathematical case, such participation does not render problematic the meeting of the criteria of individuality and plurality.

If we still accept the principle of having a set with physical objects of a set-theoretic nature, by overviewing the whole suprastructure created through the process of mathematical modelling, integrating both the empirical and mathematical structures by means of the external function (a set, in turn), we shall put in evidence so-called "pure" sets (having only mathematical objects as elements), so-called "physical" sets (having only physical objects as elements), and so-called "mixed" sets. This differentiation induces non-homogeneity of the nature of the constituent substructures, and of the suprastructure itself, with respect to their nodes.

II.1. The nature of constituent structures and the justification function of the structural metamodel

In what follows, I shall argue that this non-homogeneity of the suprastructure (with respect to either nodes or connections) created in the structural metamodel of application of mathematics does render problematic the existence and/or consistency of a metamodel's justification function.

By establishing the external function of homo/isomorphic correspondence between the source and target structures (whether postulated, built, or with proven existence) a suprastructure is created on the basis of the set-theoretic nature of the whole representation. This suprastructure does exist as a set-theoretic object, since the corresponding structures are sets, while external function is also a set (of pairs of nodes). However, this set-theoretic object also has an epistemic nature. On one hand, its components (source and target structures and external function) are constructed with the goal of acquiring new knowledge (the relation inferred on the basis of the homo/isomorphic character of the correspondence). On the other hand, the created suprastructure itself is an epistemic object, at a metatheoretical level (as an outcome/result of the metamodel), and also at the level of the global practice of applied mathematics. Indeed, once a mathematical application has been confirmed as successful, the correspondences and interpretations made within this application will be used in other new applications as acquired confirmed knowledge. Thus, we cannot ignore either the set-theoretic nature of the suprastructure (as long as the model was founded with the concepts of set theory) or its epistemic nature (closely related to the constitutive elements, as well as to the goal of the theoretical model).

However, it is just this dual nature that poses problems, in my view, to the justification function of such a theoretical model, which, besides representation, should provide a theoretical motivation for application and applicability of mathematics.

In the mathematical domain, the set-theoretic structure extracted for application has been obtained through a reduction of the Bourbaki structures participating in the relevant mathematical theory. This reduction is entirely justified, being actually a mathematical *equalization* with a structure whose relations are mathematically defined and whose nodes have an atomic status. But the Bourbaki structure has an epistemology different from its set-theoretic equivalent (this is why I initially referred to these mathematical structures as 'epistemic'). The existence of an empirical content at the intentional level of the creation of the mother-structures is one reason, since this empirical content vanishes with the set-theoretic reduction. The axiomatic method participates in the definition of the *types* of Bourbaki structures and in turn eradicates the empirical content of the axioms; however, it is just the logical functionality based on the content elimination that ensures the 'intelligibility' and unity of mathematics as a network of structures. Therefore, the epistemic suprastructure created by the structural metamodel, which integrates the set-theoretic mathematical structure and not the "original" Bourbaki one, will not carry the entire specificity of the mathematical method, despite the set-theoretic equalization. Such an epistemic inadequacy does affect the justification function of the metamodel which, without that component, seems to be one of reasoning through analogy and not necessarily through mathematical reasoning.

With regard to the nodes of the structures, the set-theoretic suprastructure will have both physical and mathematical objects as nodes, which qualifies it as a mixed set and structure. The set of physical objects, when accepted, would affect the suprastructure at the conceptual-constitutive level, and this poses a problem; besides that problem, the existence of mixed sets (either as subsets of the total set, or set-theoretic representations of certain connections between a physical and a mathematical node) presents an additional, special problem. Indeed, not having the certainty of the diversity of the individuals within a set, we will not have any guarantee for the existence of some connections already engaged in the suprastructure – that is, an identity between two elements will dissolve a possible connection between them within a certain relation. Let us observe that the only connections holding nodes of both natures (physical and mathematical) are[20] the connections belonging to the binary relation of correspondence f, namely $(x, f(x))$. In this form, these ordered pairs express the choices made for creating the theoretical conditions of the modeling, that is, the structural correspondence. Although conventional, these choices are also based on previous knowledge. (Some choices, already operated in confirmed previous applications, will be maintained as credible.) Thus, as a connection, $(x, f(x))$ has an epistemic nature which reflects the particularity of the method used to acquire knowledge. Once we express this connection set-theoretically, in the denotation $\{x, \{x, f(x)\}$, since set $\{x, f(x)\}$ is mixed, that set becomes susceptible to failing a test of diversity. Were such a thing to happen, connection $(x, f(x))$ would become senseless or vanish, and the entire epistemic construction based on an external relation between the two domains would collapse[21]. Thus, the metamodel is again affected in its justification function, for its central method itself (the structural morphism thorough conventional correspondence) is unsure or unsafe. One more time, the set-theoretic nature and the epistemic nature of the suprastructure become incompatible with respect to the metatheoretical justification.

Finally, at the level of the relations/connections in the suprastructure, let us observe that that constitutive conventionalism, which I mentioned at the beginning of section 2, is of three types: 1) In the physical structure (including the unknown connections

[20] I refer to those connections present in the suprastructure of the theoretic-applicative model of a represented arbitrary application, because other connections between the nodes of this suprastructure might exist outside the model, belonging to relations from other applications.

[21] The metamodel degenerates in this case into an 'internal relation' one.

inferred through modeling), structuring as idealization is made through the predicative-type linguistic description (in mixed language), and the relations are the result of associations operated on the basis of those predications, which are bearers of empirical knowledge, but they do not justify in any way the set-theoretic reduction; we may call this conventionalism as *atomist-idealizationant*. 2) In the mathematical (classical set-theoretic) structure, all relations are defined in mathematical language, being consistent with the deductions and definitions of mathematical theories as logical systems; the conventionalism of the relations is related only to the reduction of the Bourbaki structures; we may call it *logical* conventionality. 3) The binary relations of the external correspondence represent within the metamodel only the theoretic method that was used[22], and so we may call this conventionality *methodological*.

It is obvious that the three types of constitutional conventionality of the structures are different, and moreover, seem to be suitable for a hierarchization by the epistemic degree of set-theoretic reduction and conventional association. However, such an order translates set-theoretically as a second-order relation of the created suprastructure, which is not represented in the metamodel. Staying only with the nature of the conventionality, the detected differences revert to a difference in the epistemologies of the connections between the nodes of the suprastructure. The set-theoretic approach inevitably imposes a homogeneity of the atomic components (nodes and connections). We have already discussed the problem of the epistemic homogeneity of the nodes. Why would an epistemic homogeneity of the connections be necessary to a justification function of the metamodel? Because the justification of the method of mathematical modeling assumes that the constructed metamodel will reflect the *particularities* of this method, and these particularities revert just to the different natures of the conventionality of the structural arrangement,

---

[22] Even though I mentioned before that that correspondence of the nodes is not entirely arbitrary, being also guided by previous knowledge, this knowledge is not represented in the structural metamodel.

which is described by the nodal connections[23]. The merely set-theoretic approach cancels these particularities of the method, which are supposed to contribute to the justification of the use of the method. Let us notice that set-theoretic homogeneity does not pose problems for the representation, but only for the application of mathematics (if application of mathematics means only a procedure of reasoning through structural analogy) and not for applicability of mathematics, which cannot be described exclusively set-theoretically. Instead, the justification function of such a structural metamodel is affected by the double nature of the suprastructure it creates – set-theoretic and epistemic.

Avoiding this problem would be possible only if we drop the idea of structural integration of the two structures – source and target – through the structural morphism. By keeping the two structures separated, we would at least gain homogeneity within each of them (of their nodes, but also of their relations), and we would eliminate the problem of mixed sets. However, as I will argue further, this position would reveal another problem, that of the truth.

II.1.1. The problem of truth bearers and transfer of the truth value

To the question of whether truth should be involved in the structural models of application and applicability of mathematics, I give a positive answer which I support with two arguments. The first argument is the existence of *prediction* as a particular goal of

[23] Obviously, we have the freedom to create a mathematical representation of these differences – if possible – and to define a more complex concept of connection. For example, the notion of graph has been generalized (from applicative-theoretical necessity) to that of weighted graph, in which the edges are assigned numbers. However, the current analysis has as its object the metamodel in its current set-theoretic primary form. If one comes to a similar generalization for the connections, the structural morphism should be redefined, too, as a stronger morphism, to preserve also certain relations between the added elements.

mathematical models. There is a wide category of predictive models, whose goal (prediction) is formulated in terms of a phenomenon occurring, of the behavior of a system, or following a trajectory, etc. The second argument is the mere use of the mathematical truths from the source domain. The primary motivation for the creation and use of a mathematical model is the epistemic support we gain from mathematical necessity, which we cannot deal with directly in the target domain, which is governed by contingency. Even if we do not consider mathematical necessity as a truth[24] on the basis of which we build another truth, the motivation of the "guaranteed" epistemic support grants any belief or proposition obtained as result of the modeling the quality of being a classical truth bearer[25].

By accepting the idea of the truth of mathematical propositions and the equivalence that the set-theoretic reduction of the Bourbaki structures (seen as sets of axioms) creates, we may say – with a certain reserve – that conventional relations established in the mathematical domain are truth bearers. The same cannot be said about the relations from the target domain, the empirical one, where both the known and unknown relations are not defined, but *interpreted* (through criteria of relevance and convenience); before this interpretation, the relations have the status of a simple set-theoretic-type conventional association; it is problematic to state that a relation that is a set is a truth bearer. If in the target structure, the relation inferred through modeling (unknown prior to application) is not a truth bearer, then the final outcome of the application (after its interpretation in the empirical context) cannot be assigned a truth value. Even though interpretation through referents of the sentence is admitted as a truth condition in classical theories of truth, the interpretation of the abstract relation in a sentential form cannot stand for a truth condition because the interpretation remains in the abstract non-

[24] By adopting the classical view in which the only mathematical truth is that of the analyticity of the proof, and axioms have the same status as that of the proven result, namely non-truth-bearer (see [Hempel, 1945])

[25] But not necessarily true, in the sense of a confirmational truth

empirical realm of language[26]. One can object to this problem by saying that regardless of the way we obtain it, the final outcome of the modeling, including interpretation, is a mixed sentence, which is a traditional truth bearer. However, the issue stands not only in the existence of truth bearers in *both* domains, but also in the continuity of the process of transfer of the truth value from the source to the target domain, and this continuity is interrupted before interpretation of the conventional relations in the empirical context, as I argue above.

The difference in nature of the two truths of the modeling, namely the necessary (used) and the contingent (inferred) – assuming these do exist – apparently correlated with the quality of truth bearer or non-bearer of the corresponding entities in the two domains, does not support the opposition of the possible objection above, but potentiates it. Admitting the inferential quality of the modeling, the inference of a contingent truth from a necessary truth raises the epistemic problem of a metatheoretical explanation for the cases of empirical information of the inferred truth, an explanation which cannot be obtained outside the theoretical framework that establishes the nature and properties of the external relation between the two domains. However, this limitation is problematic, as long as truth is confirmed empirically.

Besides the difference in nature of the two truths, let us observe that the truth value can be transferred from the source domain to the target one only through the external homo/ isomorphic relation, which is the epistemic base of the inference through the modeling. Once the pure mathematical nature of this external relation is acknowledged, all its immediate or derived properties must be related to the definition of homomorphism, which is limited to the set-theoretic relational aspects, while truth has no constitutive or derived relation with this definition. Then comes the question: On the basis of what argument do we consider homomorphism as preserving not only structures, but also the truth values assigned in one of the domains?

---

[26] See also *Fig. 3*.

In conclusion, as concerns the transfer of the truth value from one domain to another, the continuity of the process of transfer seems interrupted in two places: once in the main interpretation step of the ICAM scheme (an inacceptable transfer in the formal system of the homomorphism), and once again in the interpretation in the empirical context of the relation inferred in the target structure (the absence of the quality of truth bearer of the conventional relation). The problematic processes in light of the above arguments are illustrated in the next scheme.

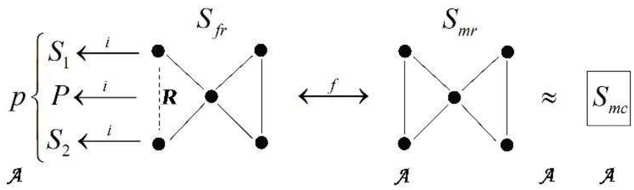

Fig. 3. Localization of truth bearers and the transfer of the truth value in the mapping account

In Figure 3, S_{mc} is the mathematical Bourbaki substructure necessary for application; S_{mr} is the relational conventional structure equivalent ($\approx$) to S_{mc}; *f* is the structural homo/isomorphism; S_{fr} is the relational physical structure; *R* is the unknown relation inferred through the model (corresponding to a known relation from S_{mr}); *i* is the process of sentential interpretation of relation *R*, assigning the subjects S_1 and S_2 to its *relata* and predicate *P* to the relation *R* (as connections); sentence $p: S_1S_2P$ is the mixed statement of the result of the double interpretation (in the physical structure and in the formulated empirical context). Marks A represent the localization of the truth bearers (traditional and non-traditional) both at the level of abstract entities (relations, propositions and sentences) and that of the processes ($\approx$) that allow an unproblematic transfer of the truth value. According to this scheme, the transfer of the truth value is interrupted before each of the two interpretations, in processes *f* and *i*.

In conclusion, even if we maintain the source and target structures as separate, not integrating them into a suprastructure, the different natures of the structures generate a new problem, that of the truth, which obviously participates in the justification function of the metamodel.

III. Conclusions

A theoretical metamodel of application and applicability of mathematics cannot ignore a justification function of its own. The set-theoretic structural approach, despite its potential of representation, cannot contribute sufficiently to a strong justification function. In this paper, I have argued that the difference in nature of the structures involved in the structural metamodels of application and applicability of mathematics poses a constitutive-type problem for the justification function. The functionality of the theoretical model assumes a commitment to the existence of a suprastructure that integrates the source and target structures as well as their external relation, and this suprastructure has also an epistemic nature. But this epistemic nature is incompatible with the set-theoretic homogeneity, with respect to both nodes and relations. If we drop the idea of suprastructure, by keeping the homogeneity of each of the corresponding structures, their different natures pose a problem of truth bearers and of the transfer of the truth value from the mathematical to the physical domain.

Eliminating these objections toward a metamodel with a valid and functional justification function seems possible only through extreme changes in theoretical and conceptual nature, which might extend even to the primary concept of *relation*. Such changes also challenge the general *representation* of the application of mathematics, bringing into discussion other available types of reasoning besides that of structural analogy.

REFERENCES

J.R.B. Arenhart, O. Bueno, "Structural Realism and the Nature of Structure", *European Journal for Philosophy of Science,* (2015) 5 (1): 111-139.

A. Baker, "The Indispensability Argument and Multiple Foundations for Mathematics", *Philosophical Quarterly*, (2003) 53: 49-67.

M. Balaguer, *Platonism and Anti-Platonism in Mathematics*, New York: Oxford University Press, 1998.

S. Bangu, *The Applicability Of Mathematics In Science: Indispensability And Ontology*, Houndmills, UK: Palgrave Macmillan, 2012.

C. Bărboianu, *The Structural Alternatives and Metatheoretical Aspects of the Solutioning of the Problem of Applicability of Mathematics in Sciences*, doctoral thesis, University of Bucharest – Department of Philosophy, 2017.

N. Bourbaki, "The Architecture of Mathematics", *The American Mathematical Monthly*, (1950) 57(4): 221-232.

O. Bueno, M. Colyvan, "An Inferential Conception of the Application of Mathematics", *Noûs*, (2011) 45(2): 345-374.

O. Bueno, S. French & J. Ladyman, "On Representing the Relationship between the Mathematical and the Empirical", *Philosophy of Science*, (2002) 69: 497–518.

N. da Costa & S. French, *Science and Partial Truth*, New York: Oxford University Press, 2003.

L. Corry, "Nicolas Bourbaki and the Concept of Mathematical Structure", *Synthese*, (1992) 92(3): 315-348.

M. Dummett, *Frege: Philosophy of Mathematics*, Cambridge, MA: Harvard University Press, 1991.

S. French, *The Structure of the World: Metaphysics and Representation*, Oxford: Oxford University Press, 2014.

S. French & J. Ladyman, "A Semantic Perspective on Idealization in Quantum Mechanics" in *Idealization in Contemporary Physics*, N. Shanks (Ed.), Amsterdam: Rodopi, 1998: 51-73.

C.G. Hempel, "Geometry and Empirical Science", *The American Mathematical Monthly*, (1945) 52(1): 7-17.

J. Ladyman, "On the Identity and Diversity of Objects in a Structure" in *Aristotelian Society Supplementary Volume*, Vol. 81, No. 1. Oxford: The Oxford University Press, 2007.

M. Leng, "What's Wrong with Indispensability? (Or the Case for Recreational Mathematics)", *Synthese*, (2002) 131: 395-417.

D. Lomas, "What perception is doing, and what it is not doing, in mathematical reasoning", *The British Journal for the Philosophy of Science*, (2002) 53(2): 205-223.

A.G. Mujumdar, T. Singh, "Cognitive science and the connection between physics and mathematics" in *Trick or Truth?* (pp. 201-217). Springer International Publishing.

J. Piaget, *Le Structuralisme*, Paris: Presses Universitaire de France, 1968.

C. Pincock, "A New Perspective on the Problem of Applying Mathematics", *Philosophia Mathematica*, (2004) 12(2): 135-161.

M.D. Resnik, *Mathematics as a Science of Patterns*, Oxford: Clarendon Press, 1997.

S. Shapiro, "Mathematics and Reality", *Philosophy of Science*, 1983, Vol. 50: 523-548.

M. Steiner, *The Applicability of Mathematics as a Philosophical Problem*, Cambridge, MA: Harvard University Press, 1998.

B. Teissier, "Protomathematics, perception and the meaning of mathematical objects" in *Images and Reasoning*, P. Grialou, G. Longo and M. Okada (Eds.), Tokyo: Keio University, 2005.

E.P. Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", *Communications on Pure and Applied Mathematics*, (1960) 13(1): 1-14.

F. Ye, "The Applicability of Mathematics as a Scientific and a Logical Problem", *Philosophia Mathematica*, (2010) 18(2): 144-165.